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Quantum mechanics and the equivalence principle

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Abstract
A quantum particle moving in a gravitational field may penetrate the classically forbidden region of the gravitational potential. This raises the question of whether the time of flight of a quantum particle in a gravitational field might deviate systematically from that of a classical particle due to tunnelling delay, representing a violation of the weak equivalence principle. I investigate this using a model quantum clock to measure the time of flight of a quantum particle in a uniform gravitational field, and show that a violation of the equivalence principle does not occur when the measurement is made far from the turning point of the classical trajectory. The results are then confirmed using the so-called dwell time definition of quantum tunnelling. I conclude with some remarks about the strong equivalence principle in quantum mechanics.

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1. Introduction

The general theory of relativity, and its plausible near-variants, is founded on the principle of equivalence of inertial and gravitational mass, a property normally associated with Galileo's experiment of dropping different masses from the Leaning Tower of Pisa. The traditional equivalence principle is fundamentally both classical and local, and it is interesting to enquire how it is to be understood in quantum mechanics. Classically, when the inertial mass m_i and the gravitational mass m_g are equated, the mass drops out of Newton's equations of motion, implying that particles of different mass with the same initial conditions follow the same trajectories. But in Schrödinger's equation the masses do not cancel. For example, in a uniform gravitational field,

$$-\left(\frac{\hbar^2}{2m_\mathrm{i}}\right)\frac{\partial^2\psi}{\partial x^2} + m_\mathrm{g} g x\psi = \frac{\mathrm{i}\hbar\partial\psi}{\partial t} \tag{1.1}$$

implying mass-dependant differences in motion. If the motion of a quantum particle is represented by a sharply peaked wave packet, then by Ehrenfest's theorem one expects that the expectation value of the particle's position will follow a geodesic, and so provide a natural classical limit that complies with the equivalence principle. There will, of course, be mass-dependant quantum fluctuations about the mean geodesic motion. This problem has been studied by Viola and Onofrio (1997).

However, one may also consider quantum states of a very different form, for example, energy eigenstates extended over a large region of space. Such quantum states do not have classical counterparts in localized bodies moving on well-defined trajectories. Rather, they might correspond to a steady flux of particles coming from a great distance. What can be said about the principle of equivalence in such a case? In general, a non-local wavefunction is able to 'feel' the spacetime curvature, and the quantum particle will respond to tidal gravitational forces (Colella *et al* 1975, Peters *et al* 1999, Speliotopoulos and Chiao 2004). But suppose one were to restrict the analysis to a *uniform* gravitational field (as in equation (1.1)), for which the spacetime curvature is zero? Might the principle of equivalence then hold even for stationary states?

At first sight the answer would seem to be no. Consider a variant of the simple Galileo experiment, where particles of different mass are projected vertically in a uniform gravitational field with a given initial velocity v. Classically, it is predicted that the particles will return a time $2v/g$ later, having risen to a height $x_{\mathrm{max}} = v^2/2g$. But quantum particles are able to tunnel into the classically forbidden region above x_{max}. Moreover, the tunnelling depth depends on the mass. One might therefore expect a small, but highly significant mass-dependant 'quantum delay' in the return time. Such a delay would represent a violation of the equivalence principle.

To investigate this scenario, it is necessary to have a clear definition of the time of flight of the quantum particle. Two problems then present themselves. First, in the case of narrow wave packets one may follow, say, the peak or the median position of the packet as it moves. But this strategy will not work for spread-out energy eigenstates. So how can one measure the time of flight of a particle between fixed points in space when its position uncertainty is very great, without collapsing the wavefunction to a position eigenstate in the process? Fortunately this problem was solved long ago by Peres (1980), who introduced a simple model quantum clock. The clock measures the *time difference* that a particle takes to travel between two points in space, without disclosing the *absolute* time of passage. This avoids collapsing the wavefunction to a position eigenstate. In effect, this device measures the phase change of the wavefunction during the particle's transit of a specified spatial region. This definition of transit time has been successfully applied to the motion of a relativistic particle (Davies 1986) and to motion across and through potential barriers (Aharonov *et al* 2002, Olkhovsky *et al* 2002, Davies 2004a, 2004b). A refinement of the Peres clock has also been studied by Alonso *et al* (2003).

The second problem is that, as the particle has a finite probability of penetrating into the classically forbidden region, this analysis involves the vexed issue of how long a particle takes to tunnel through a potential barrier (see, for example, Landauer and Martin (1994)). There is an extensive literature on this contentious topic, which I shall not attempt to summarize here. Rather, I merely remark that the Peres clock yields precise expectation values of tunnelling times that are physically very plausible in the case of square barriers (Aharonov *et al* 2002, Olkhovsky *et al* 2002, Davies 2004a, 2004b), so it seems reasonable to employ this method to discuss the 'quantum Galileo experiment.' As it happens, the main result of this quantum clock analysis is confirmed using an alternative definition of quantum transit time—the so-called dwell time—even though these two definitions do not generally produce identical results (Davies 2004a).

This topic is of more than purely theoretical interest. It is now possible to observe quantum effects of individual atoms moving in the Earth's gravitational field (see, for example, Amino *et al* (1993), Peters *et al* (1999)), and one could envisage space-based experiments using the much smaller gravitational field of, say, the International Space Station, where the quantum effects are of correspondingly greater relative size.

2. Quantum clock

The Peres clock has a single degree of freedom—rotation around a 'clock face'—and is designed to run so long as the moving quantum particle lies within a defined region of space. The expectation value of the 'hand' of the clock remains fixed once the particle has left that region, and may be read by a normal position measurement at any stage subsequently. In effect, this clock measures the change in the phase of the wavefunction, $\Delta\theta$, with the elapsed time, given by

$$\Delta T = \hbar \left(\frac{\partial}{\partial E}\right) \Delta\theta. \tag{2.1}$$

Full details are given in Peres (1980).

As an illustration of the use of the Peres clock for a simple out-and-back time measurement, consider reflection from a square potential step in one space dimension:

$$\begin{aligned} V &= 0, \qquad x < 0 \\ V &= V_0, \qquad x > 0. \end{aligned} \tag{2.2}$$

The Schrödinger equation for a uniform flux impinging on the step from the left with $E < V_0$ may readily be solved to give

$$\begin{aligned} u(x) &\propto \mathrm{e}^{\mathrm{i}kx} + A\,\mathrm{e}^{-\mathrm{i}kx}, \qquad x < 0 \\ &\propto B\,\mathrm{e}^{-px}, \qquad x > 0, \end{aligned} \tag{2.3}$$

where $k = (2mE/\hbar^2)^{1/2}$, $p = [2m(V_0 - E)/\hbar^2]^{1/2}$, and A and B are constants. Continuity of u and u' at $x = 0$ yields

$$A = -\frac{(p + \mathrm{i}k)}{(p - \mathrm{i}k)}. \tag{2.4}$$

At a point $x = -d$ from the step, the phase change between the incident (right-moving) and reflected (left-moving) wave is

$$\Delta\theta = 2kd + \arctan\left(\frac{\mathrm{Im}\,A}{\mathrm{Re}\,A}\right). \tag{2.5}$$

The latter term on the right-hand side of equation (2.5) represents the contribution from the particle tunnelling into the classically forbidden region $x > 0$. Using equation (2.1), one obtains

$$\Delta T = \frac{2d}{v} + \frac{2a}{v}, \tag{2.6}$$

where $v = \hbar k$ corresponds to the classical velocity and $a \equiv 1/p$ is a measure of the depth of penetration of the evanescent wave into the barrier. (Note that it is not necessary to determine the overall normalization factor of the wavefunction to compute ΔT; any constant phase factor will be eliminated by the differentiation in equation (2.1).) The expectation value for the time of flight therefore corresponds to the classical time for the particle to reach the barrier a distance d away at speed v, and then return at the same speed, plus an extra duration for it to tunnel beyond the classical reflection point. This 'quantum delay' in the reflection time may be written as

$$\frac{2\hbar}{[E(V_0 - E)]^{1/2}} = \left(\frac{2}{m}\right)^{1/2} \left(\frac{\hbar}{v}\right) \left(V_0 - \frac{1}{2}mv^2\right)^{-1/2} \tag{2.7}$$

which vanishes as expected in the classical limit, $\hbar \to 0$, and for an infinite step, $V_0 \to \infty$. Note that for a fixed v this expression is m-dependant.

Although it is not generally meaningful to assign a velocity to the particle beneath the barrier, in this simple example the Peres clock yields a tunnelling duration which is in fact equal to the barrier penetration depth divided by the classical speed of the incident particle.

As a second example, consider the potential

$$V = a\,\mathrm{e}^{\beta x}. \tag{2.8}$$

The Schrödinger equation may be solved for the following energy eigenstates:

$$u(x) \propto \mathrm{e}^{\pi k/\beta} K_{2\mathrm{i}k/\beta}\left(\sqrt{\left(\frac{8m\alpha}{\hbar^2\beta^2}\right)}\,\mathrm{e}^{\beta x/2}\right) \tag{2.9}$$

which are chosen to remain finite as $x \to \infty$. Evaluating equation (2.9) in the limit $x \to -\infty$, where $V \to 0$, yields

$$\pi\left(\frac{\beta}{4k}\right)\mathrm{e}^{\pi k/\beta}\,\mathrm{cosech}\left(\frac{2\pi k}{\beta}\right)\left\{\Gamma^{-1}\left(\frac{2\mathrm{i}k}{\beta}\right)\exp\left[\mathrm{i}kx + \mathrm{i}\left(\frac{k}{\beta}\right)\ln\left(\frac{2m\alpha}{\hbar^2\beta^2}\right)\right] + \mathrm{cc}\right\}. \tag{2.10}$$

The first term in the outer brackets represents a wave travelling to the right, approaching the potential hill, and the complex conjugate term represents the reflected wave. The phase change between the incident and reflected waves at $x = -X < 0$ is

$$\Delta\theta = -2kX - 2\varphi - 2\left(\frac{k}{\beta}\right)\ln\left(\frac{2m\alpha}{\hbar^2\beta^2}\right) \tag{2.11}$$

where

$$\varphi = \arctan\left[\frac{\mathrm{Im}(\Gamma^{-1})}{\mathrm{Re}(\Gamma^{-1})}\right]. \tag{2.12}$$

The phase factor φ simplifies for large β

$$\varphi \approx \arctan\left(\frac{\beta}{2kC}\right) \tag{2.13}$$

where C is Cantor's number. Using equations (2.1) and (2.11), we obtain for the out-and-back time of flight

$$\Delta T = -\frac{2X}{v} + \left(\frac{2}{\beta v}\right)\ln\left(\frac{4E}{\alpha}\right) + \left(\frac{4}{\beta v}\right)\left[-\ln\left(\frac{2mv}{\hbar\beta}\right) + C\Big/\left(1 + \frac{4C^2m^2v^2}{\hbar^2\beta^2}\right)\right] \tag{2.14}$$

where $v \equiv \hbar k/m$, this being the incident classical velocity in the asymptotic region $x \to -\infty$, where $V \to 0$. The first two terms on the right-hand side of equation (2.14) are the classical time of flight; the third term is a mass-dependant quantum correction that takes account of tunnelling. Note that this correction vanishes as $\beta \to \infty$, as expected. In this case the potential becomes a sharp wall, and the situation is identical to the previous example with $V_0 \to \infty$.

3. The gravitational case

I now consider the situation for a quantum particle moving in a uniform gravitational potential in one dimension:

$$V(x) = m_{\mathrm{g}}gx. \tag{3.1}$$

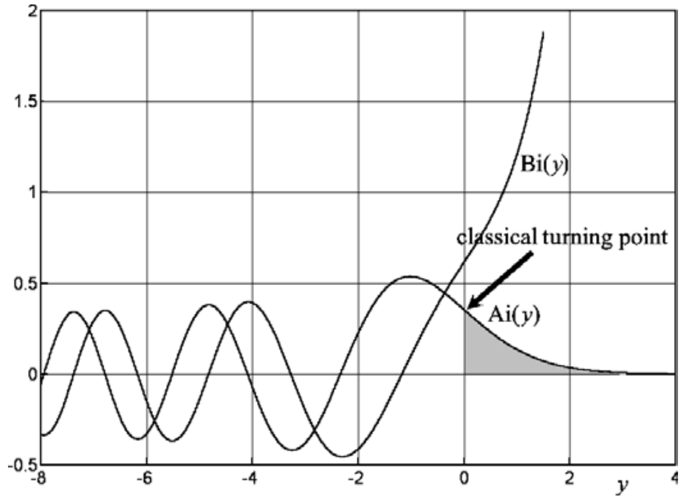

Figure 1. The real function Ai(y) ($y \equiv -z$) is bounded as $y \to \infty$, and so is chosen as the appropriate wavefunction in place of the linearly independent solution Bi(y), which is unbounded as $y \to \infty$. The point $y = 0$ corresponds to the classical turning point of the particle's trajectory. Note that Ai(y) oscillates in the region $y < 0$, and dips just before $y = 0$, indicating that there is a finite probability of the particle scattering back before reaching the classical turning point. Conversely, there is a non-zero probability that the particle will be found in the classically forbidden (shaded) region $y > 0$.

Solutions of the Schrödinger equation (1.1) that are finite for $x \to \infty$ are well known in terms of Airy functions (Bessel functions of order 1/3) (see, for example, Davies and Betts (1994)). For eigenstates of energy E, one obtains

$$u(x) \propto \mathrm{Ai}\left[\frac{(x-b)}{a}\right] \tag{3.2}$$

where

$$a = \left(\frac{\hbar^2}{2m_\mathrm{i} m_\mathrm{g} g}\right)^{1/3}, \qquad b = \frac{E}{m_\mathrm{g} g}, \tag{3.3}$$

and from here on I shall put $m_\mathrm{i} = m_\mathrm{g} \equiv m$. The solution is shown in figure 1. Note that the particle may tunnel into the classically forbidden region $x > b$, where Ai decays exponentially (evanescent wave) to a depth of order a. For an electron near the Earth's surface, this distance is about 1 mm. Near an object such as a space station, this tunnelling distance would be several orders of magnitude greater.

We wish to consider the wavefunction in the region $x < b$ where the following decomposition in terms of Bessel functions applies

$$\mathrm{Ai}(-z) = \tfrac{1}{3}\sqrt{z}[J_{1/3}(\zeta) + J_{-1/3}(\zeta)], \tag{3.4}$$

where

$$\zeta = \frac{2}{3} z^{3/2}, \qquad z = \frac{(b-x)}{a} \qquad > 0.$$

The right-hand side of equation (3.4) may be re-expressed as the linear combination

$$\tfrac{1}{3}\sqrt{z}\{[\mathrm{e}^{\mathrm{i}\pi/3} J_{1/3}(\zeta) + \mathrm{e}^{-\mathrm{i}\pi/3} J_{-1/3}(\zeta)] + [(1-\mathrm{e}^{\mathrm{i}\pi/3}) J_{1/3}(\zeta) + (1-\mathrm{e}^{-\mathrm{i}\pi/3}) J_{-1/3}(\zeta)]\}. \tag{3.5}$$

The second term in the outer parentheses is in fact just the complex conjugate of the first. The terms in the square brackets correspond to incident (up-moving) and reflected (down-moving) waves, respectively, as may be verified by computing the current vectors of each:

$$j(x) = \left(\frac{\hbar}{2\mathrm{i}m}\right)\left(u^*\frac{\mathrm{d}u}{\mathrm{d}x} - \mathrm{cc}\right) = \pm\left(\frac{N^2}{4\pi}\right)\left(\frac{2\hbar g}{m}\right)^{1/3} = \text{constant}, \qquad (3.6)$$

where N is a normalization constant, and use has been made of the Wronskian relation

$$J_{1/3}(\zeta)J'_{-1/3}(\zeta) - J_{-1/3}(\zeta)J'_{1/3}(\zeta) = \left(\frac{2}{\zeta\pi}\right)\sin\left(\frac{\pi}{3}\right). \qquad (3.7)$$

The decomposition into incident and reflected waves may also be seen explicitly by taking the large z limit of the Bessel functions:

$$\mathrm{Ai}(-z) \approx \left(\frac{1}{3\pi}\right)^{1/2} z^{-1/4}\left[\mathrm{e}^{\mathrm{i}\pi/3}\cos\left(\zeta - \frac{5\pi}{12}\right) + \mathrm{e}^{\mathrm{i}\pi/3}\cos\left(\zeta - \frac{\pi}{12}\right)\right] + \mathrm{cc}. \qquad (3.8)$$

After some work, one finds

$$\begin{aligned}\mathrm{Ai}(-z) &\approx \frac{1}{2}\pi^{-1/2}z^{-1/4}\,\mathrm{e}^{\mathrm{i}(\zeta-\pi/4)} + \frac{1}{2}\pi^{-1/2}z^{-1/4}\,\mathrm{e}^{-\mathrm{i}(\zeta-\pi/4)} \\ &\approx \pi^{-1/2}z^{-1/4}\sin\left(\zeta + \frac{\pi}{4}\right)\end{aligned} \qquad (3.9)$$

which contains incident and reflected waves of equal amplitude but different phase.

Consider now the phase change between the incident and reflected waves at $x = -X < 0$ far from the classical turning point, i.e. in the above limit of $X \to \infty$, consequent upon the reflection and propagation:

$$\Delta\theta = 2\left(\zeta + \frac{\pi}{4}\right) = \frac{4}{3}\left[\frac{(E/mg - X)}{a}\right]^{3/2} - \frac{\pi}{2}. \qquad (3.10)$$

From this one obtains, using equation (2.1), the out-and-back (or up-and-down) travel time

$$\Delta T = \left(\frac{2\hbar}{mga}\right)\left[\frac{(E/mg - X)}{a}\right]^{1/2} = 2\left[\frac{2(b-X)}{g}\right]^{1/2}. \qquad (3.11)$$

A classical particle with total energy E projected vertically at $x = -X < 0$ will climb to a distance $d \equiv b - X$ in a time $\sqrt{(2d/g)}$, so its turnaround time is

$$\Delta T_{\text{classical}} = 2\sqrt{\frac{2d}{g}}. \qquad (3.12)$$

Comparing equations (3.11) and (3.12) we see immediately that they are the *same*: the expectation value for the turnaround time of a quantum particle is identical to the classical time, when the measurement is performed far from the classical turning point. In this sense, the principle of equivalence holds for a quantum particle, even in an unbound delocalized energy eigenstate. It is remarkable that although equation (3.11) is a quantum result, $\hbar$ has cancelled from the expression for ΔT.

How can this result be understood, when we know that that the wavefunction is non-vanishing beyond the classical turning point $x = b$? Subject to the usual caveats about attempting to interpret quantum processes in classical language, a plausible explanation is apparent. Whilst there is a finite probability that a given particle may tunnel into the region above the classical turning point and return late, there is also a finite probability that the particle may backscatter off the gravitational potential before it reaches $x = b$. This is consistent with the fact that the wavefunction (hence probability density) dips prior to $x = b$ (see figure 1). This may be contrasted with the classical probability density, which rises like $(x - b)^{-1/2}$

near $x = b$ for a uniform stream of particles. It would appear that, in the case of the uniform gravitational potential equation (3.1), these two effects exactly cancel, leading to neither a shortening nor a lengthening of the classical turnaround time due to quantum effects. This interpretation will be confirmed in the next section.

The foregoing state of affairs pertains only to large distances from the classical turning point. If one were to measure the turnaround time close to $x = b$, one might expect to 'miss' those particles that have scattered back early, and to record a positive delay due to quantum penetration of the gravitational potential. To investigate this, one may compute the phase change just below the classical turning point at $x = b$. Expanding the Bessel functions for small ζ in equation (3.5) yields

$$3^{-2/3}\Gamma^{-1}\left(\tfrac{2}{3}\right) e^{i\pi/3} z \qquad \text{and} \qquad 3^{-4/3}\Gamma^{-1}\left(\tfrac{4}{3}\right) e^{-i\pi/3} \tag{3.13}$$

for the incident and reflected waves, respectively, from which the phase change is found to be

$$\Delta\theta = \arctan\left\{2\sqrt{3}\frac{[3^{2/3}\Gamma(2/3)z - 3^{4/3}\Gamma(4/3)]}{[3^{2/3}\Gamma(2/3)z + 3^{4/3}\Gamma(4/3)]}\right\}. \tag{3.14}$$

Using equations (2.1) and (3.14), and taking $z \to 0$ afterwards, gives the expected non-zero tunnelling delay at the turning point:

$$\Delta T = \frac{4.3^{-1/6}\Gamma(2/3)\hbar}{13\Gamma(4/3)mga} \approx 0.5\left(\frac{\hbar}{mg^2}\right)^{1/3}. \tag{3.15}$$

4. The dwell time

It is of interest to compare the results of the Peres clock with those using an alternative definition of the transit time. One that is much discussed is the so-called dwell time ΔT_D (Büttiker 1983) defined to be the probability P of the particle residing in the classically forbidden region divided by the incident flux (as a result of which the normalization constant cancels). A detailed discussion of the nature of quantum measurements that yield the dwell time has been given by Steinberg (1995).

The dwell time adapts naturally to the present problem. I start by computing its expectation value for a particle that is tunnelling into the classically forbidden region $x > b$, $z < 0$. Here, the Airy function may be written in terms of a MacDonald function $K_{1/3}$, whence

$$P = \left(\frac{N^2 a}{3\pi^2}\right)\int_0^\infty z K_{1/3}^2(\zeta)\,d\zeta, \tag{4.1}$$

where now $\zeta = \frac{2}{3}(-z)^{3/2}$. The integral may be evaluated:

$$P = \left(\frac{N^2 a}{4\pi^2}\right) 3^{1/3}\Gamma^2\left(\frac{2}{3}\right) \tag{4.2}$$

Dividing P by the flux given by equation (3.6) yields a value for the dwell time beneath the potential barrier of

$$\Delta T_D = \left(\frac{1}{\pi}\right)\left(\frac{3}{4}\right)^{1/3}\Gamma^2\left(\frac{2}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3} \approx 0.4\left(\frac{\hbar}{mg^2}\right)^{1/3}. \tag{4.3}$$

Comparing equations (3.15) and (4.3) we see that the dwell time has the same form and is numerically close to the expectation value of the tunnelling time as determined by the Peres clock. But, significantly, is not identical. To obtain an idea of the numbers involved, for an electron near the Earth's surface, the dwell time in the tunnelling region is about 4 ms—significantly long by the standards of atomic physics.

I now consider the opposite limit: the dwell time for a particle projected vertically a great distance from the turning point. From equations (3.2) and (3.6), one has

$$\Delta T_{\mathrm{D}} = 4\pi \left(\frac{m}{2\hbar g}\right)^{1/3} \int_X^\infty \mathrm{Ai}^2(-z)\,\mathrm{d}x$$

$$= 4\pi a \left(\frac{m}{2\hbar g}\right)^{1/3} \int_0^{(b-X)/a} \mathrm{Ai}^2(-z)\,\mathrm{d}z + \left(\frac{1}{\pi}\right)\left(\frac{3}{4}\right)^{1/3} \Gamma^2\left(\frac{2}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3} \quad (4.4)$$

using equation (4.3) for the portion of the integral in the region $z < 0$. (Note that Ai is real for real values of the argument.) The integral in equation (4.4) cannot be evaluated in terms of simple functions for arbitrary X, but it suffices for this analysis to deal with the limit of large $|X|$, for comparison with equation (3.11). To do this, first consider the approximation of replacing Ai by its asymptotic form (3.9). The integral may then be written as

$$\left(\frac{1}{2\pi}\right)\left(\frac{2}{3}\right)^{2/3} \left\{\int_0^\infty \zeta^{-2/3} \sin 2\zeta\,\mathrm{d}\zeta + \int_0 \zeta^{-2/3}\,\mathrm{d}\zeta\right\} \quad (4.5)$$

where the upper limit on the (convergent) first integral may now safely be extended to ∞, and the second (indefinite) integral may be performed explicitly. The result when substituted in equation (4.4), gives

$$2\left[\frac{2(b-X)}{g}\right]^{1/2} + 2^{-1/3}3^{-2/3}\Gamma\left(\frac{1}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3} + \left(\frac{1}{\pi}\right)\left(\frac{3}{4}\right)^{1/3}\Gamma^2\left(\frac{2}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3}. \quad (4.6)$$

The first term is recognized as the classical time, which of course diverges when $X \to -\infty$.

It is now fairly straightforward to calculate the (finite) difference between the foregoing approximate, and the exact, values of the dwell time in the region ($0 < z < -X$), in the large $-X$ limit. The details of this are given in the appendix. This correction term (given by equation (A.7)) exactly cancels the second two terms of (4.6), to yield the final result, for large $-X$

$$\Delta T_{\mathrm{D}} \to 2\left[\frac{2(b-X)}{g}\right]^{1/2} = 2\sqrt{\frac{2d}{g}} = \Delta T_{\mathrm{classical}}. \quad (4.7)$$

Thus the dwell time calculation confirms the analysis of the previous section. Once again, $\hbar$ cancels out from the transit time, and the classical result is recovered, so long as the projection point is located well away from the turning point. This concordance occurs in spite of the fact that the dwell time generally does *not* produce the same results as the Peres clock; indeed, they even differ in the current calculation in the time they assign to the tunnelling in the region $x > b$. But it all comes out in the wash in the asymptotic region, suggesting that the main result (3.11) and (4.7) is a special feature of the gravitational potential. This calculation also confirms the conjecture given in the previous section that the delay due to quantum tunnelling at the top of the trajectory is exactly compensated by the accumulated backscattering in the region below the classical turning point. To see this, one simply notes that the dwell time for the restricted region $x < b$ ($z > 0$) is equal to the classical time plus the quantum correction term $-(1/\pi)\left(\frac{3}{4}\right)^{1/3}\Gamma^2\left(\frac{2}{3}\right)(\hbar/mg^2)^{1/3}$. This correction term reduces the expectation value of the turnaround time by exactly the same amount as the dwell time for the tunnelling region $x > b$ ($z < 0$) increases it.

5. Discussion

The results of this paper suggest that a uniform gravitational potential—which applies locally to any non-singular gravitational field—has a special property in relation to quantum mechanics, namely that the expectation time for the propagation of a quantum particle in this background is identical to the classical propagation time. This may be taken as an extension of the principle of equivalence into the quantum regime (for a broader discussion of what is entailed by a 'quantum equivalence principle,' see Lämmerzahl (1996)). This special property seems to depend on the form of the potential; it does not apply in the case of a sharp potential step, or an exponential potential, for example, as is evident from the calculations presented in section 2.

The recovery of the classical transit time applies only to measurements made far from the classical turning point at $x = b$. The distance scale for this approximation is determined by the length $a = (\hbar^2/2m^2g)^{1/3}$, which roughly corresponds to one de Broglie wavelength from the turning point. Within this distance there are significant quantum corrections to the turnaround time, including the possibility of a mass-dependant delay due to the penetration of the classically forbidden region $x > b$ by the evanescent part of the wavefunction (3.2). This quantum 'smearing' of the equivalence principle is restricted to distances within the normal position uncertainty of a quantum particle. It is noteworthy, however, that even in the Earth's gravitational field the quantum corrections near the classical turning point are considerable. For an electron, the average penetration depth into the gravitational potential is about 1 mm and the corresponding delay time is several milliseconds. It is conceivable that such effects may be measurable using existing technology. In a space-based experiment, utilizing the small gravitational field of the space vehicle, the effects would be very much greater.

Because the Peres clock is itself a quantum system, it is subject to intrinsic uncertainty in its performance. There will also be a back-reaction of the operation of the clock on the measured particle (Peres 1980). These effects introduce errors in ΔT comparable to equation (3.15). Some reduction in this uncertainty may be obtained by using pulsed rather than continuous coupling between the particle and the clock (Alonso *et al* 2003), but it cannot be eliminated altogether. The back-reaction may be reduced by making the coupling between the clock and the particle arbitrarily small, but at the expense of increasing the uncertainty in the clock position. However, this irreducible uncertainty may be compensated for by introducing a large ensemble of identical systems, and regarding the time measurement as a *weak measurement* (Aharaonov *et al* 1988). It is in this ensemble sense that the times computed in this paper are to be regarded.

I have restricted attention to the so-called weak equivalence principle. One might also enquire into the status of the strong or Einstein equivalence principles in quantum mechanics. Einstein made the postulate that all of physics in a uniform gravitational field should be locally equivalent to the physics in a uniformly accelerated frame. Does it apply to the problem discussed in this paper? It is well known (see, for example, Viola and Onofrio (1997)) that under a transformation of coordinates to an accelerated reference frame

$$\begin{aligned} x' &= x - vt - \left(\tfrac{1}{2}\right) at^2 \\ t' &= t \end{aligned} \tag{5.1}$$

the Schrödinger equation for a free particle is transformed into the Schrödinger equation for a particle moving in a uniform gravitational potential with $g = a$. So there is a formal correspondence between a uniform gravitational field and a uniform acceleration in the underlying quantum kinematics, just as there is in classical kinematics. However, in quantum mechanics the relationship between the state of the system and the dynamical evolution is much more subtle than in classical mechanics. In the case of states that represent localized wave packets, the equivalence of acceleration and gravitation goes through in a reasonably straightforward manner (Viola and Onofrio 1997). But what about the case of the non-localized energy eigenstates considered in this paper? Here the situation is more complicated. The stationary states of the Hamiltonian

$$-\left(\frac{h^2}{2m}\right)\frac{\partial^2}{\partial x^2}+mgx \tag{5.2}$$

have the form

$$\mathrm{Ai}\left[\frac{(x-b)}{a}\right]\mathrm{e}^{-\mathrm{i}Et/\hbar} \tag{5.3}$$

for energy E. Now consider the situation viewed from an accelerated frame, where the particle moves freely. The appropriate energy eigenstates have the form

$$\mathrm{e}^{\mathrm{i}kx-\mathrm{i}Et/\hbar}. \tag{5.4}$$

Under the transformation (5.1), the eigenfunctions (5.4) do not transform into (5.3). The equation of motion may transform correctly, but the energy eigenstates do not. Rather, the Airy functions will be complicated linear combinations of plane wave solutions (5.4) and their complex conjugates. (The transformation of plane wave solutions into accelerated reference frames is a well-studied problem; see, for example, Birrell and Davies (1982), section 4.5.) This would not matter if the results of the analysis were linear in the wavefunction. That is indeed the case for the behaviour of wave packets which are made up of linear combinations of plane waves. But it is not the case for a measurement of the transit time, at least when such a measurement is made using the Peres clock prescription considered here. That is because the time interval depends on a measurement of the phase change, and the sum of the phases of a superposition of waves is generally not the same as the phase of the sum. A Peres clock will generally respond to a superposition of states in a very complicated way. To be sure, a stationary Peres clock will register the classical transit time $\Delta T = d/v$, where $v = \hbar k$ and d is the traversed distance, for the energy eigenstate (5.4). But one cannot use this simple fact combined with the transformation (5.1) to deduce the main result equation (3.11). To test the quantum correspondence between stationary and accelerated reference frames, it is necessary to consider the response of an *accelerated* quantum clock to the state (5.4). Given the well-known differences between the results of alternative model clocks to quantum tunnelling times, and given the small but significant difference found here between equations (3.15) and (4.3), it is far from clear that all model quantum clocks will respond equally in this scenario.

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Appendix

To evaluate the difference between the approximate integral (4.5) and the exact, but potentially divergent, integral shown in the second line of equation (4.4), first introduce a regulation parameter α (<1), and make the replacement

$$\int_0^\infty \mathrm{Ai}^2(-z)\,\mathrm{d}z \to \int_0^\infty \mathrm{Ai}(-\alpha z)\mathrm{Ai}(-z)\,\mathrm{d}z. \tag{A.1}$$

One may let $\alpha \to 1^-$ at the end of the calculation. Introducing this parameter enables the upper limit of both integrals in the approximation (4.5) to be extended to ∞, yielding

$$\left(\frac{1}{4\pi}\right)\left(\frac{2}{3}\right)^{2/3}\Gamma\left(\frac{1}{3}\right)\alpha^{-1/6}[(1+\alpha)^{-1/3}+3^{1/2}(1-\alpha)^{1/3}]. \tag{A.2}$$

The exact integral (A.1) may be performed by using the decomposition (3.4) and evaluating the products of Bessel functions and powers in terms of hypergeometric functions (Gradshteyn and Ryzhik 1965, equation (6.574)). This results in the following expression:

$$\left(\frac{3^{1/3}}{9}\right)\left[3\alpha^{2/3}\Gamma^{-2}\left(\frac{1}{3}\right)F\left(1,\frac{2}{3};\frac{4}{3};\alpha^2\right)+\Gamma^{-1}\left(\frac{2}{3}\right)F\left(\frac{1}{3},\frac{2}{3};\frac{2}{3};\alpha^2\right)\right.$$
$$\left.+\Gamma^{-1}\left(\frac{2}{3}\right)F\left(\frac{2}{3},\frac{1}{3};\frac{2}{3};\alpha^2\right)\right]. \tag{A.3}$$

Each hypergeometric function may be transformed into a function of $1-\alpha^2$ (Gradshteyn and Ryzhik, equation (9.131.2)), resulting in

$$\left(\frac{3^{1/3}}{9}\right)\left[-3\alpha^{2/3}\Gamma^{-2}\left(\frac{1}{3}\right)F\left(1,\frac{2}{3};\frac{4}{3};1-\alpha^2\right)+a^{2/3}(1-\alpha)^{-1/3}\right.$$
$$\left.\times\Gamma^{-1}\left(\frac{2}{3}\right)F\left(\frac{1}{3},\frac{2}{3};\frac{2}{3};1-\alpha^2\right)+2\Gamma^{-1}\left(\frac{2}{3}\right)(1-a)^{-1/3}\right]. \tag{A.4}$$

One may now use the hypergeomentric series to expand the functions F about $1-\alpha^2$. Discarding those terms in the series that $\to 0$ as $\alpha \to 1$, (A.4) reduces to

$$3^{-2/3}\Gamma^{-1}\left(\tfrac{2}{3}\right)(1-\alpha^2)^{-1/3}-3^{-2/3}\Gamma^{-2}\left(\tfrac{1}{3}\right). \tag{A.5}$$

We must now take the difference between the exact expression (A.5) and the asymptotic approximation (A.2). Using the relation $\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)=\pi/\cos(\pi/6)$, it can be seen that the terms which diverge as $\alpha \to 1$ cancel, leaving the desired finite correction term when $\alpha = 1$ of

$$-\left(\frac{1}{4\pi}\right)2^{1/3}3^{-2/3}\Gamma\left(\frac{1}{3}\right)-\left(\frac{1}{4\pi^2}\right)3^{1/3}\Gamma^2\left(\frac{2}{3}\right). \tag{A.6}$$

Finally, multiplying by the pre-factor of the integral in the second line of equation (4.4) and using equation (3.3) for the definition of a, we arrive at the correction for the dwell time

$$-2^{-1/3}3^{-2/3}\Gamma\left(\frac{1}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3}-\left(\frac{1}{\pi}\right)\left(\frac{3}{4}\right)^{1/3}\Gamma^2\left(\frac{2}{3}\right)\left(\frac{\hbar}{mg^2}\right)^{1/3}. \tag{A.7}$$

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