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Cosmological horizons and entropy

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Abstract. An analogue of Hawking's black hole area theorem is proved for Friedmann-type cosmological models with event horizons. The generalised second law of thermodynamics is investigated in cases where the horizon shrinks.

1. Introduction

One of the most remarkable advances in fundamental physics was the discovery by Bekenstein (1973) and Hawking (1975) that black holes represent entropy, quantified by the area of their event horizon, A. The so-called generalised second law of thermodynamics then asserts that

$$\dot{S}_{\rm bh} + \dot{S}_{\rm m} \geqslant 0 \qquad (1.1)$$

where $S_{\rm bh} = 2\pi A$ is the black hole entropy in units $\hbar = c = 8\pi G = k = 1$, and $S_{\rm m}$ is the entropy of matter in the region outside the horizon.

Black holes are, however, only one type of global structure that contain event horizons. Many cosmological models also have event horizons. The thermodynamic status of cosmological horizons remains to be clarified. Is the horizon area still a measure of entropy? Will the generalised second law continue to hold for reasonable assumptions about the matter content of the spacetime?

One cosmological case has received a lot of attention, namely, de Sitter space. Gibbons and Hawking (1977) have asserted that the generalised second law extends to de Sitter horizons, and detailed investigation (Davies 1987) confirms this.

On the other hand, the thermodynamic status of the event horizon in de Sitter space differs in a rather deep way from the black hole case. First, the thermal character of the horizon is rather subtle. An inertial particle detector in a de Sitter-invariant vacuum state responds as if it is immersed in a bath of thermal radiation of temperature

$$T_{\rm h} = H/2\pi \qquad (1.2)$$

where H is the (constant) Hubble parameter for de Sitter space (see, for example, Birrell and Davies 1982). However, unlike in the black hole case, the stress-energy-momentum tensor of this quantum state does *not* correspond to that of thermal radiation. Indeed, any thermal radiation present in de Sitter space is rapidly red-shifted away by the expansion. There is no asymptotically flat spacetime region where the thermal radiance of the horizon may be compared to that of an ordinary hot body. A related fact is that different inertial observers see differently located event horizons.

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Secondly, in the black hole case it is possible to quantify the entropy of the hole in terms of the loss of information concerning the matter that imploded to form the hole in the first place (Bekenstein 1973). This argument depends upon the existence of a well defined black hole mass energy, and the use of the first law of thermodynamics (conservation of mass energy). The horizon structure of de Sitter space is quite different. The observer is located 'inside' rather than 'outside' the horizon. The absence of asymptotic flatness precludes a meaningful definition of the mass energy of de Sitter space, and hence precludes a use of the first law.

In some sense, the amount of information hidden in the infinite volume of space behind the de Sitter horizon is infinite. This reflects the fact that there are an infinite number of non-overlapping regions of de Sitter space each containing a (different) horizon. Alternatively, however, one can try to quantify the horizon entropy around a given location by 'growing' de Sitter space from a matter-dominated cosmology in which the horizon area is negligible near $t=0$ (Davies 1988).

In the black hole case, the area A of the horizon is non-decreasing with time so long as the dominant energy condition on the stress-energy-momentum tensor $T_{\mu\nu}$ of the matter is satisfied:

$$T^{00} \geqslant |T^{\alpha\beta}| \qquad \alpha, \beta = 1, 2, 3. \tag{1.3}$$

For a perfect fluid with pressure p and energy density ρ, equation (1.3) implies

$$\rho \geqslant 0 \tag{1.4}$$

and

$$-\rho \leqslant p \leqslant \rho \tag{1.5}$$

which together imply that

$$\rho + p \geqslant 0. \tag{1.6}$$

Although physically reasonable, there are circumstances in which the dominant energy condition fails. One of these is, in fact, in the vicinity of a black hole in which the matter environment is in certain quantum vacuum states. It can then happen that $\rho < 0$, leading to a failure of (1.4). This produces the so-called Hawking effect, in which the black hole shrinks, due to the inward propagation of negative energy across its horizon.

In spite of the fact that the area theorem is violated by the Hawking effect, the combined quantity $S_{\text{bh}} + S_{\text{m}}$ remains non-decreasing. This is because thermal radiation is produced by the black hole, which 'pays' for the loss of horizon area. Thus if one regards $2\pi A$ as the entropy of the black hole, the total entropy complies with a generalised second law of thermodynamics.

It is easy to establish an analogous area theorem for cosmological event horizons in the case of small perturbations around de Sitter space (Davies 1987). Although de Sitter space possesses an associated Hawking temperature, there is no corresponding evaporation effect. However, if one relaxes the dominant energy condition, not by allowing ρ to become negative, but by allowing $p < -\rho$, then the horizon area will again shrink. One way to do this is to consider the effect of bulk viscosity. If the cosmological medium has equation of state $p = (\gamma - 1)\rho$ and bulk viscosity $\eta = \alpha\rho$ ($\alpha = \text{constant} > 0$), then the effective pressure $p' = p - 3H\alpha\rho$, where $H \equiv \dot{a}/a$ is the Hubble parameter and a is the cosmological scale factor. If $\gamma < 3H\alpha$, $p' < -\rho$ and it follows that $\dot{A} < 0$. Nevertheless, a generalised second law analogous to (1.1) is still satisfied in this case. The entropy generated by viscous effects offsets the reduction in horizon area.

It is often argued that the pressure of a fluid cannot exceed $|\rho|$ because the speed of sound would then exceed the speed of light. However, it has been suggested by Morris and Thorne (1988) that this argument does not apply to certain exotic quantum states, and that care must be taken to distinguish between group and phase velocity when the medium is dispersive. It is also worth noting that the above argument assumes $p\to 0$ as $\rho\to 0$, which need not be the case for some quantum states.

2. Area theorems for spacetimes with Robertson–Walker metrics

Consider a class of homogeneous isotropic cosmological models with Robertson-Walker metric

$$\mathrm{d}s^2=\mathrm{d}t^2-a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2}+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right) \tag{2.1}$$

where $k=0, \pm 1$, filled with a fluid of pressure p and energy density ρ. Except where stated I assume that the topology is R^4 when $k=0, -1$ and $R\times S^3$ when $k=+1$. The proper distance from the origin of coordinates to the (spherical) event horizon is defined to be

$$R_{\mathrm{h}}=-a\times\begin{cases}\sinh\eta & k=-1\\ \eta & k=0\\ \sin\eta & k=+1\end{cases} \tag{2.2}$$

where η is the conformal time difference

$$\eta=-\int_t^\infty\frac{\mathrm{d}t'}{a(t')}\qquad |\eta|<\infty. \tag{2.3}$$

If $|\eta|=\infty$ no event horizon exists. I denote by a prime and a dot, respectively, differentiation with respect to η and t.

The following theorem will now be proved.

Theorem. If the cosmological fluid is subject to condition (1.6) and $a\to\infty$ as $t\to\infty$ then the event horizon area is non-decreasing with time.

Proof. It suffices to show that $R_{\mathrm{h}}'\geqslant 0$. (Because $a(t)>0$, η is a monotonic increasing function of t so $\dot R_{\mathrm{h}}\geqslant 0$ if $R_{\mathrm{h}}'\geqslant 0$.)

The Einstein equations may be written

$$\dot\rho=-3H(\rho+p) \tag{2.4}$$

$$3H^2=\rho-3k/a^2. \tag{2.5}$$

Equations (2.4) and (2.5) yield

$$\dot H-k/a^2=-\tfrac{1}{2}(\rho+p). \tag{2.6}$$

Putting $c=a'/a$, equation (2.6) becomes

$$c'-c^2-k=-\tfrac{1}{2}a^2(\rho+p). \tag{2.7}$$

On the assumption that $\rho+p\geqslant 0$, equation (2.7) gives

$$c'-c^2-k\leqslant 0. \tag{2.8}$$

The $k=0$ case is easiest. From (2.2) one sees that $R'_{\rm h}\geqslant 0$ implies $a'\eta+a\leqslant 0$ or (noting that $\eta<0$)

$$c\geqslant -1/\eta. \tag{2.9}$$

To prove that (2.8) implies (2.9) one notes that the inequality (2.8) can be integrated with respect to η from $\eta(<0)$ to 0:

$$\int_c^{c_0}\frac{{\rm d}c}{c^2}\leqslant\int_\eta^0 {\rm d}\eta=-\eta \tag{2.10}$$

where c_0 is the value of c for $\eta=0$, i.e. $t=\infty$ (see equation (2.3)). Note that the validity of (2.10) depends upon (2.8) being true for *all* time after the moment of interest, i.e. in the interval $[\eta, 0]$.

From (2.10)

$$1/c_0-1/c\geqslant\eta \tag{2.11}$$

which implies (2.9) if (i) $c>0$ and (ii) $c_0=\infty$, i.e. $1/c_0=0$. As far as (i) is concerned, this is the condition that the universe should be expanding. (If a $k=0$ universe contracts, subject to $(\rho+p)>0$, then it will encounter a singularity, in which case the condition that $a\to\infty$ as $t\to\infty$ will be violated.) To show that $c_0=\infty$, note from (2.3) that

$$\int_t^\infty\frac{{\rm d}t'}{a(t')}=\int_a^\infty\frac{{\rm d}a}{a\dot a}=\int_a^\infty\frac{{\rm d}a}{a'}=\int_a^\infty\frac{{\rm d}a}{ac}<\infty. \tag{2.12}$$

Suppose that $1/|c_0|$ is bounded from below, i.e. $1/|c|>\varepsilon>0$ as $a\to\infty$, $t\to\infty$. Then

$$\int_a^\infty\frac{{\rm d}a}{ac}>\varepsilon\int_a^\infty\frac{{\rm d}a}{a}=\infty. \tag{2.13}$$

But (2.13) conflicts with (2.12). Therefore $\varepsilon=0$ and $c_0=\infty$.

Turning to the $k=\pm1$ cases (2.9) is replaced by

$$c\geqslant\begin{cases}-\coth\eta & k=-1\\ -\cot\eta & k=+1.\end{cases} \tag{2.14}$$

Integrating (2.8) yields

$$\begin{aligned}&\coth^{-1}c_0-\coth^{-1}c\geqslant\eta && k=-1\\ &\tan^{-1}c-\tan^{-1}c_0\geqslant\eta && k=+1.\end{aligned} \tag{2.15}$$

Putting $c_0=\infty$ in (2.15), and noting that cot and coth are piecewise monotonic decreasing functions, one recovers (2.14), and the theorem is proved.

3. Horizon behaviour

The behaviour of the event horizon has some interesting features. First, taking the equality in (2.9) and (2.14) leads to the solutions

$$a(t)=\begin{cases}\sinh(\alpha t) & k=-1\\ e^{\alpha t} & k=0\\ \cosh(\alpha t) & k=+1\end{cases} \tag{3.1}$$

which corresponds to three parametrisations of de Sitter space, for which $\rho+p=0$. In these cases $R_h=\text{constant}$.

Any deviation from $\rho+p=0$ will cause the horizon area to change. The results of § 2 demonstrate that for $\rho+p>0$ the horizon must grow with time. If the universe model starts out with a big bang singularity ($a\to 0$ as $t\to 0$) then $R_h=0$ at $t=0$. The horizon starts out with zero area and then grows.

On the other hand, the existence of the $\cosh(\alpha t)$ de Sitter solution (3.1) suggests that for $k=+1$ there may quite generally exist non-singular cosmological models which involve a contraction from $a=\infty$ to some minimum value, and then a 'bounce' out again. These models have infinite age in the past, and the question arises as to how the horizon originates.

An explicit example is

$$a(t)=1+(t/t_0)^2 \qquad t_0>\sqrt{2} \tag{3.2}$$

for which $\rho+p>0$ for all t. Integrating (3.2) one finds

$$R_h=-\left[1+\left(\frac{t}{t_0}\right)^2\right]\sin\left[\frac{\pi t_0}{2}-t_0\tan^{-1}\left(\frac{t}{t_0}\right)\right] \tag{3.3}$$

$$=-\sin\eta\,\operatorname{cosec}^2(\eta/t_0). \tag{3.4}$$

Not only does (3.3) have periods when it decreases as a function of t, it can even take negative values! For example, putting $t_0=2$, (3.3) reduces to

$$R_h=t \tag{3.5}$$

which is negative for $t<0$. What has gone wrong?

The answer is that $k=+1$ Robertson–Walker models have compact spatial sections, which can make light paths multivalued. For large values of $|\eta|$, light might pass several times around the universe. From the causal point of view, a surface from which a photon arrives at $r=0$ at $t=+\infty$ having already passed $r=0$ on a previous circuit does not constitute a true horizon. To eliminate this, one must restrict $|\eta|<\pi$. The duration $|\Delta\eta|=\pi$ is the time required for a photon to reach $r=0$ from the antipodal point. (Although it requires $|\Delta\eta|=2\pi$ for a complete circumnavigation of the universe by a photon, another photon could still reach $r=0$ in a time $|\Delta\eta|<\pi$ by travelling in the opposite direction.) With the restriction $|\eta|<\pi$, (3.3) will always be a monotonic increasing function. Note that the de Sitter case corresponds precisely to the range $-\pi\leqslant\eta\leqslant 0$.

In these cases, then, no horizon exists at early times ($\eta<-\pi$). The horizon forms at $\eta=-\pi$ from a zero radius ($R_h=0$), but it forms, not about $r=0$ as in the big bang models, but about the *antipodal point* $r=\pi$. It then grows steadily in area.

By symmetry, if a horizon in a Robertson–Walker model universe grows continuously from zero radius, it must appear either at the position of the observer, or at the antipodal point. There remains, however, a curious third possibility—the abrupt appearance of a horizon at a non-zero radius.

An example of this possibility is the $k=+1$ de Sitter universe with antipodal points identified. The scale factor still behaves as in (3.1), but there is no horizon for $t<0$.

The horizon appears suddenly at $t=0$ with finite radius α^{-1}. It is interesting to note that a particle detector in such a universe still registers a continuous thermal response.

A curious feature about cosmological event horizons is their teleological quality. The horizon area is defined in terms of an integral over all future time (see (2.3)), yet it has a well defined value at each epoch t. The non-decreasing nature of the horizon area depends upon the condition $\rho+p\geqslant 0$ holding *for all future time.* Thus, if at some epoch in the far future $\rho+p<0$, this could cause the horizon area to decrease at the *current* epoch.

Similar ideas apply to black hole horizons. The position of the horizon depends on the entire future circumstances of the hole. However, in the black hole case, the hole rapidly settles down to a quasiequilibrium state in which there exists an apparent horizon, and it is this apparent horizon that plays the thermodynamic role. (The situation is analogous to a body that rapidly settles down to thermodynamic equilibrium at some temperature T, even though at some future time the temperature may change.)

However, the cosmological horizons being discussed here (with the exception of de Sitter space and perturbations around it) do *not* rapidly settle down to a quasiequilibrium state, but go on evolving for all time. The peculiar teleological quality of these horizons is thus much more pronounced. The absence of a quasiequilibrium state is, of course, also manifested in the absence of a well defined Hawking temperature for such horizons. Their thermodynamic significance is therefore much less clear than for either black hole or de Sitter spaces.

4. Contracting models

The proof of area increase for $k=0$ models given in § 2 is violated if $c<0$, i.e. if the universe contracts to a final singularity. In such cases the upper limit of the integral in (2.3) is chosen to correspond to the singularity $a=0$, in which case $\eta=0$ implies $a=0$. For small η, the difference between $k=\pm 1$ and $k=0$ in (2.2) is negligible. Hence all three classes of model exhibit horizon shrinkage in the case that they approach final singularities. Clearly the horizon radius $R_{\mathrm{h}}\approx -a\eta\to 0$.

There exist such models, in which the entropy of the cosmological fluid per comoving volume remains unchanged during the approach to the singularity (e.g. radiation filled, $k=+1$). Thus the matter entropy per horizon volume also decreases with time. There is thus a clear violation of the generalised second law.

These models evade the increasing area theorem because of the presence of a terminating singularity at $a=0$. However, the horizon area might start to decrease long before. Thus, in a perfectly normal region of spacetime the generalised second law of thermodynamics fails. Should one be alarmed by this?

There would seem to be two possible responses. It could be argued that the horizon, being a teleological entity, has no local significance anyway, except in the case where its area settles down to a fixed or slowly varying value (e.g. the approach to de Sitter space). The physically repugnant consequences of a violation of the second law have to do with the construction of a *perpetuum mobile*, etc, involving the manipulation of energy and entropy exchange in such quasistatic cases. This may not be possible when the horizon is rapidly shrinking towards a singularity a finite time in the future.

On the other hand, the failure of the generalised second law might be regarded as evidence, either that the event horizon area is no longer a suitable measure of entropy, or that it is, but that it must be augmented by an additional term. In the longstanding search (Penrose 1979) for the elusive 'gravitational entropy' the general tendency of self-gravitating systems to grow more clumpy with time is often cited as a manifestation of the growth of entropy. One could therefore attribute an entropy to the 'degree of shrinkage' of the universe. This quantity, suitably defined, would rise without limit as the singularity is approached. Thus might an extended generalised second law be satisfied.

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