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LETTER TO THE EDITOR

Cosmological horizons and the generalised second law of thermodynamics

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Abstract. I investigate the change in event-horizon area in cosmological models that depart slightly from de Sitter space. If the gravitational source is a perfect inviscid fluid satisfying the dominant energy condition, the area is proved to be non-decreasing. If the fluid is viscous, the area can decrease, but the attendant generation of entropy ensures that the generalised second law of thermodynamics remains unviolated.

Recent interest in the inflationary universe scenario has focused attention on the role of the event horizon and associated Hawking temperature of de Sitter space. Some thermodynamic properties of de Sitter horizons and black hole-de Sitter spaces have been investigated in Davies (1984) and Davies *et al* (1986).

In the real universe, the inflationary phase has to be joined on smoothly to some conventional Friedmann-like behaviour. In addition, quantum vacuum effects would generate small departures from the idealised de Sitter form. Thus a realistic inflationary model does not involve a space that is *exactly* de Sitter space. In this letter I consider Robertson-Walker cosmological models in which the scale factor departs slightly from the de Sitter form, and investigate their thermodynamic properties.

The generalised second law of thermodynamics was initially formulated for black holes (Bekenstein 1973). It states that the sum of the ordinary entropy plus one quarter of the area, A, of the hole's event horizon cannot decrease with time. (I use units in which $8\pi G=\hbar=c=k=1$.) Thus $2\pi A$ is identified with the 'gravitational entropy' of the black hole. It is widely assumed that one can associate a gravitational entropy with de Sitter space too, using the same formula $2\pi A$, where A is now the area of the de Sitter horizon (Gibbons and Hawking 1977). An analysis of the exchange of energy and entropy between heat baths and black hole or de Sitter horizons supports these ideas (Hawking 1976, Unruh and Wald 1982, Davies 1984).

It seems reasonable to extend the generalised second law to other cosmological event horizons. As a first step I discuss small departures from de Sitter space.

Consider a spatially flat Robertson-Walker model with scale factor $a(t)=\exp(Ht)$. In the case that $H=\text{constant}$, this corresponds to de Sitter space. The event-horizon area A is $4\pi/H^2$. If H is a slowly varying function of time t, then the horizon area will also change with time:

$$\dot{A}=-8\pi\dot{H}/H^3. \tag{1}$$

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To investigate this change I suppose that the source in the gravitational field equations has energy density ρ and pressure p satisfying the perfect-fluid condition

$$p=(\gamma-1)\rho \tag{2}$$

where γ is a constant.

The gravitational field equations (Friedmann equations) yield

$$3H^2=\rho \tag{3}$$

and

$$\dot{\rho}+3H(\rho+p)=0. \tag{4}$$

Equations (2)-(4) give

$$\dot{H}=-\gamma\rho/2. \tag{5}$$

The case $\gamma=0$ corresponds to de Sitter space, for which $H=\text{constant}$. More generally one obtains from (1)

$$\dot{A}\geqslant 0 \qquad \text{if } \gamma\geqslant 0 \tag{6}$$

which corresponds to the so-called dominant energy condition (Hawking and Ellis 1973). I thus arrive at the result that, subject to this condition, small departures from de Sitter space involve only *increases* in the horizon area. In some sense, then, de Sitter space represents a minimal-horizon cosmological spacetime, associated with the extremal nature of the associated equation of state of its gravitational source content, $p=-\rho$ $(\gamma=0)$.

More interesting is the case where viscosity is present. The consequences of this for the inflationary universe have recently been considered by Barrow (1987), Padmanabhan and Chitre (1987) and Modak (1987). I suppose that the viscous matter produces a perturbation around de Sitter space, so that the total energy density and pressure are the sum of a de Sitter contribution (1) and a viscous matter contribution (2): $\rho=\rho_1+\rho_2$, $p=p_1+p_2$, where

$$p_1=-\rho_1 \tag{7}$$

and

$$p_2=(\gamma-1)\rho_2. \tag{8}$$

Following Barrow, suppose the bulk viscosity coefficient $\eta=\alpha\rho$, where α is a constant greater than 0 and $\rho_2>0$. (This might result from some quantum vacuum.) Then the total pressure is given by

$$p'=p_1+p_2-3H\alpha\rho_2 \tag{9}$$

and (4) is replaced by

$$\dot{\rho}+3H(\rho+p')=0. \tag{10}$$

After some simple algebra using (3) and (10), one discovers that

$$\dot{A}=4\pi(\gamma-3H\alpha)\rho_2/H^3. \tag{11}$$

From this result it is seen that the event-horizon area will be non-decreasing as long as $\gamma>3H\alpha$. If $\gamma<3H\alpha$, the horizon shrinks with time. This does not, however, necessarily signal a violation of the generalised second law. The same viscosity that gravitationally drives the horizon to smaller values also unavoidably generates ordinary entropy. To calculate the rise in ordinary entropy I use the first law of thermodynamics

$$T\,\mathrm{d}S=\mathrm{d}E+p\,\mathrm{d}V. \tag{12}$$

Adapted to an expanding universe, (12) becomes

$$T\,\mathrm{d}S=\mathrm{d}(\rho a^3)+p\,\mathrm{d}a^3. \tag{13}$$

From equations (7)-(10) and (13) one finds

$$T\dot{S}=9H^2\alpha\rho_2 a^3. \tag{14}$$

According to the generalised second law

$$2\pi\dot{A}+\dot{S}\geqslant 0. \tag{15}$$

Substitution from (11) and (14) shows that this will be satisfied if

$$\frac{8\pi^2(\gamma-3H\alpha)}{H^3}+\frac{9H^2\alpha a^3}{T}\geqslant 0. \tag{16}$$

This will be the case if

$$T\leqslant 3a^3H^4/8\pi^2 \tag{17}$$

$$\leqslant H/2\pi \tag{18}$$

where I have taken the volume a^3 to be that within the de Sitter event horizon, $a^3=4\pi/3H^3$. But $H/2\pi$ is precisely the (irreducible) Hawking temperature for de Sitter space. If (as seems reasonable) one assumes $T\geqslant H/2\pi$, then the inequality is just satisfied.

One can regard the situation with $T=H/2\pi$ as one of thermodynamic equilibrium: the temperature of the matter is constant and exactly matched to that of de Sitter space. Suppose, however, that the temperature of the viscous matter were higher than the Hawking temperature. If $T=b(H/2\pi)$ and $\gamma<3H\alpha(1-1/b)$, then (15) would indeed be violated.

It should be noted, however, that in this case we are no longer dealing with thermodynamic equilibrium. The temperature of the cosmological material will decline as the universe expands. The attendant loss of energy will cause a back reaction on the cosmic dynamics that will serve to increase the horizon area. One might expect the rise in entropy associated with this increase to more than offset the decrease due to the other processes.

Although a general demonstration of this might be difficult, it is easy to prove for the special case that the universe is uniformly filled with thermal radiation. (I now ignore viscous effects.) In the horizon volume $4\pi/3H^3$ the total entropy of the radiation at time t is

$$S_{\mathrm{r}}=16\pi\sigma T_0^3\,\mathrm{e}^{-3Ht}/9H^3 \tag{19}$$

where σ is the radiation constant and the temperature of the radiation, $T=T_0\,\mathrm{e}^{-3Ht}$, declines with time as the universe expands. (This is equivalent to the flow across the horizon viewed as at a fixed distance $1/H$ from a given observer.) The rate of change of radiation entropy is thus

$$\dot{S}_{\mathrm{r}}=-16\pi\sigma T_0^3\,\mathrm{e}^{-3Ht}/3H^2. \tag{20}$$

The corresponding increase in horizon area brought about by the decline in radiation energy may be obtained from (1) and (5):

$$2\pi\dot{A} = 8\pi^2\gamma\rho/H^3 = 32\pi^2\sigma T_0^4\,\mathrm{e}^{-4Ht}/3H^3. \tag{21}$$

Conformity with the generalised second law of thermodynamics requires that the magnitude of (21) exceed that of (20). This will be the case if

$$T > H/2\pi \tag{22}$$

i.e. the Hawking temperature. But by assumption T is indeed greater than the Hawking temperature.

The foregoing analysis is concerned with small departures from de Sitter space. I shall report elsewhere on the status of the generalised second law for the case of event horizons in more general expanding universe models.

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