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Thermodynamic phase transitions of Kerr–Newman black holes in de Sitter space

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Abstract. A curious feature of black holes which rotate and/or carry electric charge is that as the rotation and/or charge is increased the black hole undergoes a second-order phase transition in which its specific heat changes from negative to positive. Here this phase transition is investigated for black holes in de Sitter-like universes.

1. Introduction

A generic feature of self-gravitating systems is that their specific heats are negative, that is, they tend to become hotter if they lose energy. (For a recent discussion see Pavón and Landsberg (1988).) A famous special example is the Schwarzschild black hole. According to Hawking (1975) the entropy and temperature of the hole are:

$$S = \pi r_{\mathrm{h}}^2 = 4\pi M^2 \tag{1.1}$$

$$T = \kappa/2\pi = 1/8\pi M \tag{1.2}$$

respectively, where r_{h} is the radius of the event horizon, κ is its surface gravity, M is the mass of the black hole and I have used units $\hbar = c = G = k = 1$.

It then follows from (1.1) and (1.2) that:

$$\partial S/\partial T = -64\pi M^3 < 0. \tag{1.3}$$

Curiously, however, if the black hole possesses angular momentum a and/or electric charge Q, there exist parameter ranges for a and Q for which $\partial S/\partial T > 0$ (Davies 1978). The transition from negative to positive values of $\partial S/\partial T$ occurs via an infinite discontinuity, characteristic of a second-order thermodynamic phase transition. Precisely what this transition implies for the black hole's structure is unclear.

In this paper I shall generalise the earlier work on black-hole phase transitions to include a cosmological constant Λ and, consequently, a cosmological event horizon too. The thermodynamic status of cosmological horizons, which is less clear than for black-hole horizons, is discussed in Davies (1988a, b, c).

The starting point for this generalisation is the metric of Carter (1973) for a class of black-hole solutions to the Einstein–Maxwell equations with cosmological constant Λ:

$$\begin{aligned} \mathrm{d}s^2 = {} & \rho^{-2}\chi^{-4}(\Delta_\theta a^2 \sin^2\theta - \Delta)\,\mathrm{d}t^2 + \rho^2\Delta^{-1}\,\mathrm{d}r^2 + \rho^2\Delta_\theta^{-1}\,\mathrm{d}\theta^2 \\ & + \rho^{-2}\chi^{-4}[\Delta_\theta(r^2+a^2)^2\sin^2\theta - \Delta a^2\sin^4\theta]\,\mathrm{d}\phi^2 \\ & -2\rho^{-2}\chi^{-4}a\sin^2\theta[\Delta_\theta(r^2+a^2) - \Delta]\,\mathrm{d}t\,\mathrm{d}\phi \end{aligned} \tag{1.4}$$

where

$$\Delta = (a^2 + r^2)(1 - \Lambda r^2) - 2Mr + Q^2 \tag{1.5}$$

$$\Delta_\theta = 1 + \Lambda a^2 \cos^2 \theta \tag{1.6}$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta \tag{1.7}$$

$$\chi^2 = 1 + \Lambda a^2 \tag{1.8}$$

and I have made the substitution $\Lambda \rightarrow 3\Lambda$ for simplicity of notation.

In the limit $\Lambda \rightarrow 0$, (1.4) reduces to the well known Kerr–Newman class of black-hole metrics representing a black hole with angular momentum a and electric charge Q in an asymptotically flat axisymmetric stationary spacetime. If desired one can include the effect of a magnetic monopole charge P by making the substitution $Q^2 \rightarrow Q^2 + P^2$ throughout. In the limit M, a, $Q \rightarrow 0$ (1.4) reduces to the metric for de Sitter space.

Horizons occur where $\Delta = 0$. This is a quartic equation. In descending order of their magnitude the three roots r_1, r_2, r_3 correspond respectively to the cosmological (de Sitter) horizon, the outer black-hole horizon (event horizon) and the inner black-hole horizon (Cauchy horizon). The final root r_4 is negative and does not have immediate physical significance.

The entropy associated with an event horizon is given by one quarter of the surface area of the horizon concerned, i.e.

$$\tfrac{1}{4} \int (g_{\theta\theta} g_{\phi\phi})^{1/2} \, d\theta \, d\phi \tag{1.9}$$

evaluated at the relevant root r_1 or r_2. A simple integration using (1.4) and the condition $\Delta = 0$ gives

$$S = \pi \chi^{-2} (r_h^2 + a^2) \tag{1.10}$$

with $r_h = r_1$ or r_2. Note that this result is independent of Q.

The temperature of the horizon is still given by $\kappa / 2\pi$. The surface gravities κ of the horizons of (1.4) have been evaluated by Mellor and Moss (1989)

$$\kappa_i = \tfrac{1}{2} \Lambda \chi^{-2} (r_i^2 + a^2)^{-1} \prod_{i \neq j} |r_i - r_j| \tag{1.11}$$

for the horizon corresponding to the root r_i.

2. Black-hole specific heat

In this section I shall focus on the thermodynamic phase transition of the black-hole horizon located at $r = r_2$. The specific heat at constant Λ, a and Q is defined as:

$$C_{\Lambda,a,Q} = T(\partial S / \partial T)_{\Lambda,a,Q}. \tag{2.1}$$

From (1.10) and (1.11) one has

$$\begin{aligned} C_{\Lambda,a,Q} &= T(\partial S / \partial r_h)_{\Lambda,a,Q} / (\partial T / \partial r_h)_{\Lambda,a,Q} \\ &= \frac{2\pi \kappa_2 r_h}{\chi^2 (\partial \kappa_2 / \partial r_h)_{\Lambda,a,Q}}. \end{aligned} \tag{2.2}$$

The quantities κ_2, r_h and χ^{-2} are non-vanishing, but it may happen that

$$(\partial \kappa_2 / \partial r_h)_{\Lambda,a,Q} = 0 \tag{2.3}$$

for some set of values of the parameters Λ, a, Q, in which case $C_{\Lambda,a,Q}$ passes through an infinite discontinuity, characteristic of a second-order phase transition. To locate the phase boundary in the space of the parameters Λ, a, Q one must differentiate (1.11) with respect to $r_h \equiv r_2$ and substitute into (2.3). Before this can be done, it is necessary to eliminate r_1, r_3, r_4 and M from (1.11), which may be accomplished by comparing coefficients in the quartic defining the roots:

$$\Delta = -\Lambda \prod_i (r - r_i). \tag{2.4}$$

After some algebra one obtains:

$$\kappa_2 = \tfrac{1}{2}\Lambda\chi^{-2}(r_2^2 + a^2)^{-1}[3r_2^3 - (1 - \Lambda a^2)r_2/\Lambda + (a^2 + Q^2)/\Lambda r_2]. \tag{2.5}$$

Differentiating (2.5) with respect to r_2 and substituting into (2.3) gives:

$$3\Lambda r_2^6 + (8a^2\Lambda + 1)r_2^4 + (\Lambda a^4 - 4a^2 - 3Q^2)r_2^2 - a^2(a^2 + Q^2) = 0. \tag{2.6}$$

Equation (2.6) together with the condition for the roots $\Delta = 0$ can in principle be used to eliminate r_2 and yield an equation connecting a, Q and Λ.

It is convenient to scale out the mass M and define:

$$\alpha = \Lambda M^2 \tag{2.7}$$

$$\beta = a^2/M^2 \tag{2.8}$$

$$\gamma = Q^2/M^2 \tag{2.9}$$

$$r = r_2/M. \tag{2.10}$$

The task is then to solve simultaneously the equations:

$$3\alpha r^6 + (1 + 8\alpha\beta)r^4 + (\alpha\beta^2 - 4\beta - 3\gamma)r^2 - \beta(\beta + \gamma) = 0 \tag{2.11}$$

$$\beta = (2r - \gamma)/(1 - \alpha r^2) - r^2. \tag{2.12}$$

After some further algebra the equations (2.11) and (2.12) can be recast in the form:

$$2(1 + \alpha\beta)r^3 - 3r^2 - \beta = 0 \tag{2.13}$$

(which is independent of γ). The required solution of (2.11) and (2.13) is then:

$$\alpha = r^{-3}[r - \tfrac{3}{2} + \gamma/2r - f(r)] \tag{2.14}$$

$$\beta = r^2(\tfrac{1}{2} - \gamma/2r - f(r))/(\tfrac{3}{2} - \gamma/2r + f(r)) \tag{2.15}$$

where

$$f(r) = [\tfrac{13}{4} - 2r + \gamma^2/4r^2 - 2\gamma/r + \gamma]^{1/2}. \tag{2.16}$$

In the limit $\alpha \to 0$ (i.e. $\Lambda \to 0$) the cosmological horizon no longer exists: the solution corresponds to the case of a black hole in asymptotically flat spacetime. In this case r may be explicitly eliminated to give:

$$(\beta + \gamma)^3 + \beta^2 - \beta - \tfrac{3}{4}\gamma^2 = 0. \tag{2.17}$$

For $\beta = 0$ (i.e. $a = 0$), equation (2.17) has the solution $\gamma = \frac{3}{4}$ or:

$$Q^2 = \tfrac{3}{4}M^2. \tag{2.18}$$

For $\gamma = 0$ (i.e. $Q = 0$), equation (2.17) may be solved to give $\beta = (\sqrt{5} - 1)/2$ or

$$a^2 = (\sqrt{5} - 1)M^2/2 \simeq 0.62M^2. \tag{2.19}$$

These were the results first reported for the black-hole case in Davies (1979).

3. Non-rotating case

Returning to the general case ($\Lambda \neq 0$), equations (2.14) and (2.15) simplify considerably if $\beta = 0$ ($a = 0$). One finds $\gamma = (\frac{9}{4})^2\alpha + \frac{3}{4}$, or

$$Q^2/M^2 = (\tfrac{9}{4})^2\Lambda M^2 + \tfrac{3}{4} \tag{3.1}$$

and

$$r_2 = \tfrac{3}{2}M. \tag{3.2}$$

It is curious that the radius of the black-hole horizon at the phase transition is determined entirely by M. From (3.1) it is clear that a positive value of Λ requires a greater value of Q^2 than (2.18) before the specific heat switches from negative to positive.

The parameter range in this solution is restricted by the requirement that the black hole does not become a naked singularity. This occurs when the outer and inner black-hole horizons coincide, i.e. $r_2 = r_3$, or $\kappa = 0$, in which case:

$$r_2 = \tfrac{3}{2}M - (\tfrac{9}{4}M^2 - 2Q^2)^{1/2}. \tag{3.3}$$

A comparison of (3.2) and (3.3) reveals that, to avoid a naked singularity, one requires:

$$Q^2/M^2 < \tfrac{9}{8} \tag{3.4}$$

or

$$\Lambda < 2/27M^2 \tag{3.5}$$

in (3.1). Figure 1 shows the phase boundary for this system in the α, γ plane. Also marked on the diagram are the parameter values corresponding to the Mellor–Moss instanton solution (Mellor and Moss 1989), defined by $M^2 = Q^2$, $\kappa_1 = \kappa_2$, in which the black hole and cosmological horizons are in thermodynamic equilibrium.

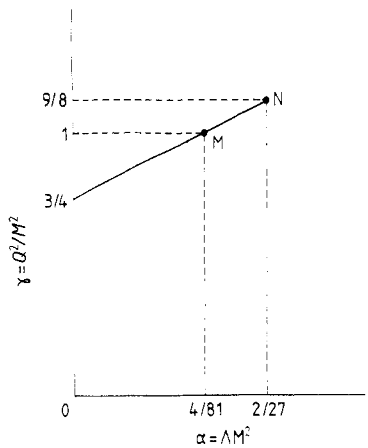

Figure 1. The full line shows the thermodynamic phase boundary for a non-rotating charged black hole in de Sitter space. The region above (below) the line corresponds to positive (negative) specific heat. The terminal point N corresponds to a naked singularity. Also marked (as M) is the Mellor–Moss instanton solution for $a = 0$. The horizontal scale of the figure is ten times the vertical scale.

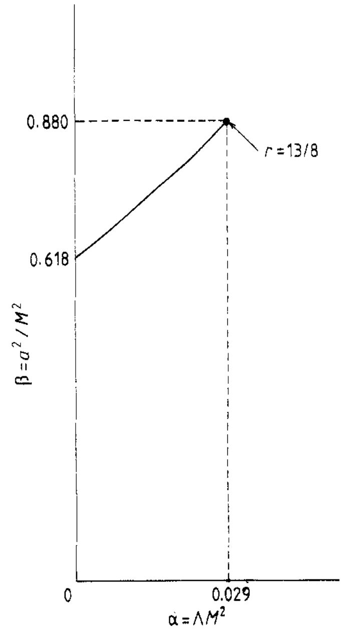

Figure 2. The full line shows the thermodynamic phase boundary for an uncharged rotating black hole in de Sitter space. The region above (below) the line corresponds to positive (negative) specific heat. The solution curve runs from $\beta=(\sqrt{5}-1)/2\simeq 0.618$ to $\frac{1}{3}(\frac{13}{8})^2\simeq 0.880$, at which point $r=\frac{13}{8}$.

4. Uncharged case

If $Q=0$, then $\gamma=0$, and (2.14) and (2.15) simplify slightly:

$$\alpha = r^{-3}[r-\tfrac{3}{2}-(\tfrac{13}{4}-2r)^{1/2}] \tag{4.1}$$

$$\beta = r^2[\tfrac{1}{2}-(\tfrac{13}{4}-2r)^{1/2}]/[\tfrac{3}{2}+(\tfrac{13}{4}-2r)^{1/2}]. \tag{4.2}$$

The parameter range of interest is

$$\left.\begin{aligned} &0\leqslant\alpha\leqslant 8^2/13^3\\ &\tfrac{1}{2}(\sqrt{5}-1)\leqslant\beta\leqslant\tfrac{1}{3}(\tfrac{13}{8})^2\\ &\tfrac{1}{2}\sqrt{5}+1\leqslant r\leqslant\tfrac{13}{8}\end{aligned}\right\}. \tag{4.3}$$

The black hole becomes a naked singularity when:

$$r^2=1-\alpha\beta-[(\alpha^2\beta^2-14\alpha\beta+1)]^{1/2}/6\alpha. \tag{4.4}$$

Examination of the limit $\alpha=0$ reveals that to avoid a naked singularity r^2 must be greater than the right-hand side of (4.4). This is the case throughout the range (4.3).

In figure 2, α is plotted as a function of β. There is a class of Mellor–Moss instanton solutions in this case too, defined by $M^2=a^2\chi^4$, $\kappa_1=\kappa_2$. However, these parameters nowhere intersect the allowed portion of the curve in figure 2.

5. Solutions without black holes

If one puts $M=0$ in (1.4) and (1.5) then the solution possesses a cosmological horizon, but no black-hole horizon. Consider first the case $a = 0$. The metric is

$$ds^2 = (1 + Q^2/r^2 - \Lambda r^2)\, dt^2 - (1 + Q^2/r^2 - \Lambda r^2)^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{5.1}$$

The spacetime is singular at $r = 0$, where it possesses a point electric charge $\pm Q$. The cosmological horizon has radius r_h and entropy

$$S = \pi r_h^2 = (\pi/2\Lambda)[1 - (1 + 4\Lambda Q^2)^{1/2}] \tag{5.2}$$

and temperature:

$$T \equiv \kappa/2\pi = (Q^2/r_h^3 + \Lambda r_h)/2\pi. \tag{5.3}$$

Straightforward differentiation reveals that $(\partial S/\partial T)_{Q,\Lambda} = 0$ when

$$Q^2\Lambda = \tfrac{3}{4}. \tag{5.4}$$

The specific heat is positive for $Q^2\Lambda > \frac{3}{4}$ and negative for $Q^2\Lambda < \frac{3}{4}$.

Now consider the case $Q = 0$, $a \neq 0$. We have:

$$S = \pi\chi^{-2}(r_h^2 + a^2) = \pi/\Lambda \tag{5.5}$$

$$T = \Lambda^{1/2}(2\pi\chi)^{-2}(r_h^2 + a^2)^{-1} \tag{5.6}$$

$$r_h = \Lambda^{-1/2} \tag{5.7}$$

whence the phase transition at $(\partial S/\partial T)_{a,\Lambda} = 0$ occurs when:

$$\Lambda a^2 = 3. \tag{5.8}$$

For $\Lambda a^2 < 3$ the specific heat is negative, for $\Lambda a^2 > 3$ it is positive.

The existence of a phase transition in this case is curious, because Carter (1989) has shown that the metric (1.4) with $M = Q = 0$ is merely de Sitter space in a rotating coordinate system. However, in considering the thermodynamic properties of the de Sitter horizon one must take into account the quantum state as well as the gravitational field. Modes of the quantum field associated with the coordinates used in (1.4) will effectively represent a 'rotating quantum vacuum', which has interesting implications for Mach's principle. I shall report on this topic elsewhere.

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References

Carter B 1973 *Black Holes* ed B DeWitt and C DeWitt (London: Gordon and Breach)
—— 1989 *Class. Quantum Grav.* **6** 569
Davies P C W 1979 *Rep. Prog. Phys.* **41** 1313
—— 1988a *Ann. Inst. Henri Poincaré* **49** 297
—— 1988b *Class. Quantum Grav.* **5** 1349
—— 1988c *Int. J. Theor. Phys.* in press
Hawking S W 1975 *Commun. Math. Phys.* **43** 199
Mellor F A and Moss I G 1989 *Class. Quantum Grav.* **6** 1379
Pavón D and Landsberg P T 1988 *Gen. Rel. Grav.* **20** 457